

# On the Dispersion Law of the Form $\varepsilon(p) = \hbar^2 p^2 / 2m + \tilde{V}(p) - \tilde{V}(0)$ for Elementary Excitations of a Nonideal Fermi Gas in the Pair Interaction Approximation ( $i \leftrightarrow j$ ), $V(|x_i - x_j|)$

V. P. Maslov\*

## Abstract

We give a derivation of the dispersion law  $\varepsilon(p) = \hbar^2 p^2 / 2m + \tilde{V}(p) - \tilde{V}(0)$ , where  $\tilde{V}(p)$  is the Fourier transform of the pair interaction potential  $V(r)$ . (The interaction between particles  $x_i$  and  $x_j$  is  $V(|x_i - x_j|)$ .)

Keywords: dispersion law, elementary excitation, pair interaction, Fermi gas.

*Dedicated to the memory of my dear student, physicist  
Vladimir Vladimirovich Belov*

## 1 INTRODUCTION

In the famous paper [1], Bogolyubov obtained the spectrum (energy levels) of a nonideal Bose–Einstein gas.

The formula for the spectrum reads

$$\lambda_p = -\hbar^2 v p + \varepsilon(p), \quad \varepsilon(p) = \sqrt{\left(\frac{\hbar^2 p^2}{2m} + \tilde{V}(p)\right)^2 - \tilde{V}^2(p)}. \quad (1)$$

where  $v$  is the fluid (gas) velocity and  $\tilde{V}(p)$  is the Fourier transform of the pair interaction potential  $V(|x_i - x_j|)$  for particles  $i$  and  $j$ .

For small  $p$ , the term  $\varepsilon(p)$  behaves as  $|p|\sqrt{\tilde{V}(0)}$ , which corresponds to sound (phonons), provided that  $\tilde{V}(0) > 0$ . The last condition can be rewritten as  $\int_0^\infty V(r) dr$  and means that the repulsion for small  $r$  exceeds the attraction occurring in helium for large  $r$ . The Fourier transform  $\tilde{V}(p)$  tends to zero as  $p \rightarrow \infty$ , and the leading term is of the order of  $p^2/2$ . The point  $p_0$  of minimum of the radicand, where the  $p$ -derivative vanishes, is called the roton part of the spectrum  $\varepsilon(p)$ .

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\*Moscow State University, v.p.maslov@mail.ru

In the same journal issue where Bogolyubov's paper was published, Landau gave a general argument based on experiments and showing that the curve  $\varepsilon(p)$  should have exactly this general form, i.e., be linear near  $p = 0$ , have one point of maximum and then a point of minimum, and then tend to infinity.

This curve is called the Landau curve (the 1962 Nobel prize).

The superfluidity of helium-4, which is a Bose gas, was discovered by Kapitsa in 1930 (the 1978 Nobel prize).

However, while the superfluid state in the Bose case is caused by short-range repulsion, it is known that the superfluid state has to be caused by long-range attraction, which enables the formation of Cooper pairs (the 1972 Nobel prize).

The superfluidity of a Fermi fluid (helium-3) was discovered experimentally in the early 1970s by the American scientists David Lee, Douglas Osheroff, and Robert Richardson (the 1996 Nobel prize).

Landau wrote a year after publishing his result: "Recently, Bogolyubov has succeeded in determining the energy spectrum of a Bose-Einstein gas with weak interaction between particles in the general case by an ingenious application of second quantization" [2, p. 43].

Specifically, Bogolyubov used Dirac's idea: since the number of particles is large, one can deem the commutator of the creation and annihilation operators in the Bose statistics to be small. The so-called Bogolyubov asymptotic  $(u, v)$ -transform leads to the same result.

In the Fermi statistics, one considers the anticommutator of the creation-annihilation operators, so that the above argument fails to provide passage to "c-numbers." This passage is only possible for pairs of fermions.

The theory of ultrasecond quantization developed by the present author permits one to obtain an even simpler formula, based on long-range attraction, for a Fermi gas. Namely,

$$\lambda_p = -\hbar^2 p v + \left| \frac{\hbar^2 p^2}{2m} + \tilde{V}(p) - \tilde{V}(0) \right|. \quad (2)$$

This formula is the subject of the present paper.

The Landau curve also arises in this case, and the main role is played by the long-range attraction provided by the interaction potential. To show this, we first study the behavior of  $\tilde{V}(p)$ .

**Remark .** The second-quantized Schrödinger equation *identically* coincides with the  $N$ -particle Schrödinger equation up to a unitary transformation. We mean that the second-quantized equation is a convenient representation of the  $N$ -dimensional system of Schrödinger equations. This representation (like the  $p$ -representation, the interaction representation, etc.) is "identical" up to a unitary transformation. It contains no approximations and is, in this sense, an identity transformation. In just the same way, ultrasecond quantization, introduced by the author, coincides up to a unitary transformation with the  $N$ -particle Schrödinger equation for symmetric and antisymmetric solutions. For some reason, this new identity scares off theoretical physicists so much that they

begin to cross themselves, saying “keep away from me,” and publicly burn the author’s books as well as his likeness.

However, Bogolyubov’s asymptotic  $(u, v)$ -transform, which is valid for bosons, coincides in the fermionic case with the ultrasecond quantization method only for Bardeen’s interaction potential. This transformation is approximate rather than exact and gives a wrong pair interaction asymptotics. Apparently, it is only after a trap experiment with a Fermi gas, similar to the one carried out in the remarkable paper [3], is conducted for the ultrasecond quantization method that theoretical physicists will believe in the method.

## 2 “GENERAL POSITION” THEOREM

In mathematics, the notion of “general position” for points of a surface, say, with respect to projection onto a coordinate plane, is well known and was in particular used in an essential way by Arnold, Varchenko, and Gusein-Zade. In physics, this special case of general position means in particular that the optical focus of a lens refracting light rays so that they all converge at a same point is not a generic situation—once the lens is moved slightly, the focus spreads into a small light spot formed by a network of intersecting caustics.

On the plane, a perpendicular drawn to the abscissa axis is not in general position—after a small rotation of the coordinate axes, this line is no longer projected into a single point on the abscissa axis. If the first two derivatives are zero at a point of a smooth curve, then the curve is not in general position with respect to a small rotation of the axes. One can rotate the axes in such a way that, in a majority of cases, the smooth curve will not have such points. Then one speaks of a curve in “general position.”

As to a point where the derivative is zero, a small rotation destroys this property. Then one speaks of a “generic” point of a smooth curve. The derivative at a generic point of a smooth curve is nonzero. In particular, the derivative of a generic smooth curve on the half-line is nonzero at the origin. If this is not the case, then a small rotation of the coordinate axes destroys this property. Thus, a small rotation does not affect the property of the derivative to be nonzero at the origin. The following elementary theorem holds.

**Theorem .** *Let the original pair interaction potential  $U(x-y)$ ,  $x, y \in \mathbb{R}^3$ , be radially symmetric and equal to  $V(r)$ , where  $r = \sqrt{\sum_{i=1}^3 (x_i - y_i)^2}$  and  $\lim_{r \rightarrow 0} V(r)r \neq 0$ . Suppose that the three-dimensional Fourier transform  $\tilde{V}(|p|)$ ,  $|p| \geq 0$ , is in general position at the origin  $|p| = 0$ . Then*

$$\lim_{p \rightarrow \infty} p^2 \tilde{V}(p) \neq 0, \quad \lim_{r \rightarrow \infty} r^4 V(r) \neq 0. \quad (3)$$

**Proof .** Since  $V(\xi)$  is spherically symmetric, it follows that so is the three-dimensional Fourier transform  $\tilde{V}(\xi)$ .

By passing to the spherical coordinates, we obtain the relation

$$V(r) = \frac{1}{r} \int_{-\infty}^{\infty} \tilde{V}(\xi) \sin(\xi r) \xi d\xi. \quad (4)$$

Since  $\tilde{V}(|\xi|)$  tends to infinity at the rate of  $\text{const}/|\xi|^2$ , it follows that

$$V(r) = \frac{\text{const}}{r} + O(1)$$

as  $r \rightarrow 0$ . The converse is also true.

The condition of general position implies that

$$\tilde{V}'(0) \neq 0.$$

One can readily see that

$$\tilde{V}'(0) = -\frac{1}{4\pi} \lim_{r \rightarrow \infty} r^4 V(r). \quad (5)$$

Indeed, by replacing the integral (4) by the integral over the finite interval  $[-A, A]$ , where  $A$  is sufficiently large, by making the change of variables  $\xi r = \eta$ , by integrating by parts, and by passing to the limit as  $r \rightarrow \infty$  and  $A \rightarrow \infty$ , we arrive at the assertion of the theorem.  $\square$

In fact, the interaction potential is not exactly spherically symmetric in view of the complicated structure of the molecule. In principle, one should take into account the influence of all particles forming the molecule. If, nevertheless, we treat the molecule as a whole and approximate the interaction potential by a spherically symmetric function, we should at least assume that the potential is “generic.”

**Corollary .** *One has*

$$\tilde{V}(p) - \tilde{V}(0) \sim -4\pi|p| \lim_{r \rightarrow \infty} r^4 V(r). \quad (6)$$

as  $|p| \rightarrow 0$ . Since the potential  $V(r)$  is attractive as  $r \rightarrow \infty$ , it follows that

$$-4\pi \lim_{r \rightarrow \infty} r^4 V(r) = c > 0. \quad (7)$$

As  $p \rightarrow \infty$ , the expression under the modulus sign in (2) tends to infinity at the rate of  $\hbar^2 p^2 / 2m$ , since  $\tilde{V}(p) \rightarrow \infty$  at the rate of  $1/|p|^2$ .

Thus, we see that formula (2) gives the Landau curve for a Fermi fluid.

**Remark .** Consider the Lennard–Jones potential and add a small smooth radially symmetric potential to it. The probability that the new potential still decays at the rate of  $1/r^6$  is zero with respect to the entire set of potentials. With probability 1, the decay will be at the rate of  $1/r^4$ . There is no known physical law that would prohibit this, and the existence of such a law is unlikely.

### 3 ULTRASECOND QUANTIZATION

Ultrasecond quantization for problems in quantum mechanics and statistical physics was introduced in [4]–[7]. Recall the notation and main facts for the case of quantization with respect to “particle–number” pairs and with respect to pairs of particles. Quantization with regard to pairs will permit us to take into account pair correlations of particles when constructing the asymptotics. The ultrasecond quantization space is the bosonic Fock space  $\mathcal{F}$  [8]. Let  $\hat{b}^+(x, s)$  be the creation operator for particles with number  $s$ , and let  $\hat{b}^-(x, s)$  be the annihilation operator for particles with number  $s$  in  $\mathcal{F}$  [8]. Next, let  $\hat{B}^+(x, x')$  be the creation operator for a pair of particles, and let  $\hat{B}^-(x, x')$  be the annihilation operator for a pair of particles in this space. These operators satisfy the fermionic anticommutation relations

$$\begin{aligned} \{\hat{b}^-(x, s), \hat{b}^+(x', s')\} &= \delta_{ss'} \delta(x - x'), & \{\hat{b}^\pm(x, s), \hat{b}^\pm(x', s')\} &= 0, \\ [\hat{B}^-(x_1, x_2), \hat{B}^+(x'_1, x'_2)] &= \delta(x_1 - x'_1) \delta(x_2 - x'_2), & [\hat{B}^\pm(x_1, x_2), \hat{B}^\pm(x'_1, x'_2)] &= 0, \\ [\hat{b}^\pm(x, s), \hat{B}^\pm(x'_1, x'_2)] &= [\hat{b}^\pm(x, s), \hat{B}^\mp(x'_1, x'_2)] = 0. \end{aligned} \quad (8)$$

For bosons,  $b^-$  and  $b^+$  satisfy similar commutation relations.

Next,  $\Phi_0$  is the vacuum vector in  $\mathcal{F}$  with the following properties:

$$\hat{b}^-(x, s)\Phi_0 = 0, \quad \hat{B}^-(x_1, x_2)\Phi_0 = 0. \quad (9)$$

The variable  $x$  ranges over the three-dimensional torus  $L_1 \times L_2 \times L_3$ , which will be denoted by  $\mathbf{T}$ . The variable  $s = 0, 1, \dots$  is discrete; it is called the *number*, or the *statistical spin*. Each vector  $\Phi \in \mathcal{F}$  can be uniquely represented in the form

$$\begin{aligned} \Phi &= \sum_{k=0}^{\infty} \sum_{M=0}^{\infty} \frac{1}{k!M!} \sum_{s_1=0}^{\infty} \cdots \sum_{s_k=0}^{\infty} \int \cdots \int dx_1 \cdots dx_k dy_1 \cdots dy_{2M} \\ &\quad \times \Phi_{k,M}(x_1, s_1; \dots; x_k, s_k; y_1, y_2; \dots; y_{2M-1}, y_{2M}) \\ &\quad \times \hat{b}^+(x_1, s_1) \cdots \hat{b}^+(x_k, s_k) \hat{B}^+(y_1, y_2) \cdots \hat{B}^+(y_{2M-1}, y_{2M}) \Phi_0, \end{aligned} \quad (10)$$

where the function  $\Phi_{k,M}(x_1, s_1; \dots; x_k, s_k; y_1, y_2; \dots; y_{2M-1}, y_{2M})$  is symmetric with respect to transpositions of the pairs  $(x_j, s_j)$  and  $(x_i, s_i)$  as well as of the pairs  $(y_{2j-1}, y_{2j})$  and  $(y_{2i-1}, y_{2i})$ . In the bosonic case, one introduces the subspace  $\mathcal{F}_{k,M}^{\text{Symm}}$  of vectors  $\Phi$  such that  $\Phi_{k',M'} = 0$  for  $(k', M') \neq (k, M)$  and  $\Phi_{k,M}$  is a symmetric function of the variables  $x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_{2M}$ . In the fermionic case, one in a similar way introduces the subspace  $\mathcal{F}_{k,M}^{\text{Asymm}}$  of vectors  $\Phi$  such that  $\Phi_{k',M'} = 0$  for  $(k', M') \neq (k, M)$  and  $\Phi_{k,M}$  is an antisymmetric function of the variables  $x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_{2M}$ . The orthogonal projec-

tion onto  $\mathcal{F}_{k,M}^{\text{Symm}}$  in  $\mathcal{F}$  has the form [4]–[7]

$$\begin{aligned}\hat{\Pi}_{k,M}^{\text{Symm}} &= \frac{1}{k!M!} \sum_{s_1=0}^{\infty} \cdots \sum_{s_k=0}^{\infty} \int \cdots \int dx_1 \cdots dx_k dy_1 \cdots dy_{2M} \\ &\quad \times \hat{b}^+(x_1, s_1) \cdots \hat{b}^+(x_k, s_k) \hat{B}^+(y_1, y_2) \cdots \hat{B}^+(y_{2M-1}, y_{2M}) \\ &\quad \times \text{Symm}_{x_1 \dots x_k y_1 \dots y_{2M}} (\hat{b}^-(x_1, s_1) \cdots \hat{b}^-(x_k, s_k) \\ &\quad \times \hat{B}^-(y_1, y_2) \cdots \hat{B}^-(y_{2M-1}, y_{2M})) \\ &\quad \times \exp \left( - \sum_{s=0}^{\infty} \int dx \hat{b}^+(x, s) \hat{b}^-(x, s) - \iint dy dy' \hat{B}^+(y, y') \hat{B}^-(y, y') \right),\end{aligned}\tag{11}$$

where  $\text{Symm}_{x_1 \dots x_k y_1 \dots y_{2M}}$  is the operator of symmetrization with respect to the variables  $x_1, \dots, x_k, y_1, \dots, y_{2M}$  and the operators  $\hat{b}^+(x, s)$ ,  $\hat{b}^-(x, s)$ ,  $\hat{B}^+(y, y')$ , and  $\hat{B}^-(y, y')$  are Wick order [8]. The orthogonal projection onto  $\mathcal{F}_{k,M}^{\text{Asymm}}$  in  $\mathcal{F}$  has the form [4]

$$\begin{aligned}\hat{\Pi}_{k,M}^{\text{Asymm}} &= \frac{1}{k!M!} \sum_{s_1=0}^{\infty} \cdots \sum_{s_k=0}^{\infty} \int \cdots \int dx_1 \cdots dx_k dy_1 \cdots dy_{2M} \\ &\quad \times \hat{b}^+(x_1, s_1) \cdots \hat{b}^+(x_k, s_k) \hat{B}^+(y_1, y_2) \cdots \hat{B}^+(y_{2M-1}, y_{2M}) \\ &\quad \times \text{Asymm}_{x_1 \dots x_k y_1 \dots y_{2M}} (\hat{b}^-(x_1, s_1) \cdots \hat{b}^-(x_k, s_k) \\ &\quad \times \hat{B}^-(y_1, y_2) \cdots \hat{B}^-(y_{2M-1}, y_{2M})) \\ &\quad \times \exp \left( - \sum_{s=0}^{\infty} \int dx \hat{b}^+(x, s) \hat{b}^-(x, s) - \iint dy dy' \hat{B}^+(y, y') \hat{B}^-(y, y') \right),\end{aligned}$$

where  $\text{Asymm}_{x_1 \dots x_k y_1 \dots y_{2M}}$  is the operator of antisymmetrization with respect to the variables  $x_1, \dots, x_k, y_1, \dots, y_{2M}$ . From now on, the operators  $\hat{b}^+(x, s)$ ,  $\hat{b}^-(x, s)$ ,  $\hat{B}^+(y, y')$ , and  $\hat{B}^-(y, y')$  are Wick ordered unless specified otherwise.

In what follows, we consider a system of  $N$  identical particles on the torus  $\mathbf{T}$ . We assume that the Hamiltonian of  $N$  bosons or fermions has the form

$$\hat{H}_N = -\frac{\hbar^2}{2m} \sum_{j=1}^N \Delta_j + \sum_{j=1}^N \sum_{l=j+1}^N V(x_j - x_l).\tag{12}$$

According to [4], the ultrasecond-quantized Hamiltonian corresponding to this

operator in the bosonic case has the form

$$\begin{aligned}
\overline{\widehat{H}}_B = & \sum_{k=0}^{\infty} \sum_{M=0}^{\infty} \frac{1}{k!M!} \sum_{s_1=0}^{\infty} \cdots \sum_{s_k=0}^{\infty} \int \cdots \int dx_1 \dots dx_k dy_1 \dots dy_{2M} \\
& \times \widehat{b}^+(x_1, s_1) \cdots \widehat{b}^+(x_k, s_k) \widehat{B}^+(y_1, y_2) \cdots \widehat{B}^+(y_{2M-1}, y_{2M}) \widehat{H}_{k+2M} \\
& \times \text{Symm}_{x_1 \dots x_k y_1 \dots y_{2M}} (\widehat{b}^-(x_1, s_1) \cdots \widehat{b}^-(x_k, s_k) \\
& \quad \times \widehat{B}^-(y_1, y_2) \cdots \widehat{B}^-(y_{2M-1}, y_{2M})) \\
& \times \exp \left( - \sum_{s=0}^{\infty} \int dx \widehat{b}^+(x, s) \widehat{b}^-(x, s) - \iint dy dy' \widehat{B}^+(y, y') \widehat{B}^-(y, y') \right).
\end{aligned} \tag{13}$$

In the fermionic case, the corresponding operator  $\overline{\widehat{H}}_F$  is expressed by a similar formula with Symm replaced by Asymm. By analogy with (12) and (13), we assign an ultrasecond-quantized operator  $\overline{\widehat{A}}$  [4] to every  $N$ -particle operator

$$\widehat{A}_N \left( x_1^2, \dots, x_N^2; -i \frac{\partial}{\partial x_1}, \dots, -i \frac{\partial}{\partial x_N} \right).$$

For example, the ultrasecond-quantized identity operator corresponding to the identity operator in the bosonic case has the form

$$\begin{aligned}
\overline{\widehat{E}}_B = & \sum_{k=0}^{\infty} \sum_{M=0}^{\infty} \frac{1}{k!M!} \sum_{s_1=0}^{\infty} \cdots \sum_{s_k=0}^{\infty} \int \cdots \int dx_1 \dots dx_k dy_1 \dots dy_{2M} \\
& \times \widehat{b}^+(x_1, s_1) \cdots \widehat{b}^+(x_k, s_k) \widehat{B}^+(y_1, y_2) \cdots \widehat{B}^+(y_{2M-1}, y_{2M}) \\
& \times \text{Symm}_{x_1 \dots x_k y_1 \dots y_{2M}} (\widehat{b}^-(x_1, s_1) \cdots \widehat{b}^-(x_k, s_k) \\
& \quad \times \widehat{B}^-(y_1, y_2) \cdots \widehat{B}^-(y_{2M-1}, y_{2M})) \\
& \times \exp \left( - \sum_{s=0}^{\infty} \int dx \widehat{b}^+(x, s) \widehat{b}^-(x, s) - \iint dy dy' \widehat{B}^+(y, y') \widehat{B}^-(y, y') \right),
\end{aligned} \tag{14}$$

which is the sum of the projections (11). Likewise, the ultrasecond-quantized identity operator in the fermionic case is given by

$$\overline{\widehat{E}}_F = \sum_{k=0}^{\infty} \sum_{M=0}^{\infty} \widehat{\Pi}_{k,M}^{\text{Asymm}},$$

and Symm in (14) is replaced by Asymm.

Consider the eigenvalue problem

$$\overline{\widehat{H}}_{B,F} \Phi = \lambda \overline{\widehat{E}}_B \Phi, \quad \overline{\widehat{E}} \Phi \neq 0, \tag{15}$$

in the bosonic and fermionic cases. The following assertion was proved in [4]. *On the subspaces  $\mathcal{F}_{k,M}^{\text{Symm}}$  and  $\mathcal{F}_{k,M}^{\text{Asymm}}$  of the space  $\mathcal{F}$ , the operators  $\widehat{H}_B$  and  $\widehat{H}_F$ , respectively, acted upon by the projection onto the  $N$ -particle space (of the operator (21) of the number of particles) coincide with the operator  $\widehat{H}_N$ . Hence the eigenvalues  $\lambda$  of problem (15) in the bosonic and fermionic cases coincide with the corresponding eigenvalues of the operators  $\widehat{H}_N$  (12). If the commutators of the operators  $\widehat{b}^-(x, s)$  and  $\widehat{b}^+(x, s)$ , as well as of  $\widehat{B}^-(x, y)$  and  $\widehat{B}^+(x, y)$ , are small of the order of  $1/N$ , then, according to [4], the asymptotics of solutions of problem (15) is determined by the symbol corresponding to problem (15). In the bosonic case, the pseudosymbol is defined in the form*

$$\begin{aligned} \mathcal{H}_B[b^*(\cdot), b(\cdot), B^*(\cdot), B(\cdot)] &= \left\{ \sum_{k,M=0}^{\infty} \frac{1}{k!M!} \sum_{s_1=0}^{\infty} \cdots \sum_{s_k=0}^{\infty} \int \cdots \int dx_1 \dots dx_k dy_1 \dots dy_{2M} \right. \\ &\quad \times b^*(x_1, s_1) \cdots b^*(x_k, s_k) B^*(y_1, y_2) \cdots B^*(y_{2M-1}, y_{2M}) H_{k+2M} \\ &\quad \times \text{Symm}_{x_1 \dots x_k y_1 \dots y_{2M}} (b(x_1, s_1) \cdots b(x_k, s_k) \\ &\quad \times B(y_1, y_2) \cdots B(y_{2M-1}, y_{2M})) \Big\} \\ &\times \left\{ \sum_{k',M'=0}^{\infty} \frac{1}{k'!M'!} \sum_{s'_1=0}^{\infty} \cdots \sum_{s'_{k'}=0}^{\infty} \int \cdots \int dx'_1 \dots dx'_{k'} dy'_1 \dots dy'_{2M'} \right. \\ &\quad \times b^*(x'_1, s'_1) \cdots b^*(x'_{k'}, s'_{k'}) B^*(y'_1, y'_2) \cdots B^*(y'_{2M'-1}, y'_{2M'}) \\ &\quad \times \text{Symm}_{x'_1 \dots x'_{k'} y'_1 \dots y'_{2M'}} (b(x'_1, s'_1) \cdots b(x'_{k'}, s'_{k'}) \\ &\quad \times B(y'_1, y'_2) \cdots B(y'_{2M'-1}, y'_{2M'})) \Big\}. \end{aligned} \quad (16)$$

The expression for the symbol in the fermionic case is similar except that Symm in (16) is replaced by Asymm. The following identity holds for the pseudosymbol (16) in the bosonic case:

$$\mathcal{H}_B[b^*(\cdot), b(\cdot), B^*(\cdot), B(\cdot)] = \frac{\text{Sp}(\widehat{\rho}_B \widehat{H})}{\text{Sp}(\widehat{\rho}_B)}, \quad (17)$$

where  $\widehat{H}$  and  $\widehat{\rho}_B$  are the second quantized operators,

$$\widehat{H} = \int dx \widehat{\psi}^+(x) \left( -\frac{\hbar^2}{2m} \Delta \right) \widehat{\psi}^-(x) + \frac{1}{2} \iint dx dy V(x, y) \widehat{\psi}^+(y) \widehat{\psi}^+(x) \widehat{\psi}^-(y) \widehat{\psi}^-(x). \quad (18)$$



Here  $\hat{\rho}_B$  depends on the functions  $b(x, s)$  and  $B(y, y')$ :

$$\begin{aligned} \hat{\rho}_B = & \sum_{k=0}^{\infty} \sum_{M=0}^{\infty} \frac{1}{k!M!(k+2M)!} \left( \sum_{s=0}^{\infty} \iint dx dx' b(x, s) b^*(x', s) \hat{\psi}^+(x) \hat{\psi}^-(x') \right)^k \\ & \times \left( \iint dy_1 dy_2 B(y_1, y_2) \hat{\psi}^+(y_1) \hat{\psi}^+(y_2) \right)^M \\ & \times \left( \iint dy'_1 dy'_2 B(y'_1, y'_2) \hat{\psi}^-(y'_1) \hat{\psi}^-(y'_2) \right)^M \exp \left( - \int dz \hat{\psi}^+(z) \hat{\psi}^-(z) \right), \end{aligned} \quad (19)$$

where  $\hat{\psi}^+(x)$  and  $\hat{\psi}^-(x)$  are Wick ordered bosonic creation–annihilation operators [7]. In the fermionic case, one has the similar identity

$$\mathcal{H}_F[b^*(\cdot), b(\cdot), B^*(\cdot), B(\cdot)] = \frac{\text{Sp}(\hat{\rho}_F \hat{H})}{\text{Sp}(\hat{\rho}_F)},$$

where  $\hat{H}$  and  $\hat{\rho}_F$  are the following second-quantized operators:

$$\begin{aligned} \hat{H} = & \int dx \hat{\psi}^+(x) \left( -\frac{\hbar^2}{2m} \Delta \right) \hat{\psi}^-(x) + \frac{1}{2} \iint dx dy V(x, y) \hat{\psi}^+(x) \hat{\psi}^+(y) \hat{\psi}^-(y) \hat{\psi}^-(x) \\ \hat{\rho}_F = & \sum_{k=0}^{\infty} \sum_{M=0}^{\infty} \frac{1}{k!M!(k+2M)!} \left( \iint dy_1 dy_2 B(y_1, y_2) \hat{\psi}^+(y_1) \hat{\psi}^+(y_2) \right)^M \\ & \times \sum_{s_1=0}^{\infty} \cdots \sum_{s_k=0}^{\infty} \int \cdots \int dx_1 dx'_1 \cdots dx_k dx'_k \\ & \times b(x_1, s_1) b^*(x'_1, s_1) \cdots b(x_k, s_k) b^*(x'_k, s_k) \\ & \times \hat{\psi}^+(x_1) \cdots \hat{\psi}^+(x_k) \hat{P}_0 \hat{\psi}^-(x'_k) \cdots \hat{\psi}^-(x'_1) \\ & \times \left( \iint dy'_1 dy'_2 B(y'_1, y'_2) \hat{\psi}^-(y'_2) \hat{\psi}^-(y'_1) \right)^M. \end{aligned} \quad (20)$$

Here  $\hat{\psi}^+(x)$  and  $\hat{\psi}^-(x)$  are the fermionic creation–annihilation operators, and  $\hat{P}_0$  is the projection onto the vacuum vector of the fermionic Fock space. In general, the pseudosymbol of the ultrasecond-quantized operator  $\widetilde{\hat{L}}$  corresponding to an arbitrary second-quantized operator  $\hat{L}$  is expressed by the formula [4]

$$L_{B,F}[b^*(\cdot), b(\cdot), B^*(\cdot), B(\cdot)] = \frac{\text{Sp}(\hat{\rho}_{B,F} \hat{A})}{\text{Sp}(\hat{\rho}_{B,F})}.$$

In the space  $\mathcal{F}$ , one introduces the ultrasecond-quantized particle number operators [4]

$$\widetilde{\hat{N}}_B = \sum_{k=0}^{\infty} \sum_{M=0}^{\infty} (k+2M) \hat{\Pi}_{k,M}^{\text{Symm}}, \quad \widetilde{\hat{N}}_F = \sum_{k=0}^{\infty} \sum_{M=0}^{\infty} (k+2M) \hat{\Pi}_{k,M}^{\text{Asymm}}. \quad (21)$$

Accordingly, the pseudosymbol of the operator  $\overline{\widehat{N}}_B$  in the bosonic case has the form

$$\begin{aligned}
N_B = & \left\{ \sum_{k=0}^{\infty} \sum_{M=0}^{\infty} \frac{k+2M}{k!M!} \sum_{s_1=0}^{\infty} \cdots \sum_{s_k=0}^{\infty} \int \cdots \int dx_1 \dots dx_{k+2M} \right. \\
& \times b^*(x_1, s_1) \cdots b^*(x_k, s_k) B^*(x_{k+1}, x_{k+2}) \cdots B^*(x_{k+2M-1}, x_{k+2M}) \\
& \times \text{Symm}_{x_1 \dots x_{k+2M}} (b(x_1, s_1) \cdots b(x_k, s_k) \\
& \quad \times B(x_{k+1}, x_{k+2}) \cdots B(x_{k+2M-1}, x_{k+2M})) \Big\} \\
& \times \left\{ \sum_{k'=0}^{\infty} \sum_{M'=0}^{\infty} \frac{1}{k'!M'!} \sum_{s'_1=0}^{\infty} \cdots \sum_{s'_{k'}=0}^{\infty} \int \cdots \int dz_1 \dots dz_{k'+2M'} \right. \\
& \times b^*(z_1, s'_1) \cdots b^*(z_{k'}, s'_{k'}) B^*(z_{k'+1}, z_{k'+2}) \cdots B^*(z_{k'+2M'-1}, z_{k'+2M'}) \\
& \times \text{Symm}_{z_1 \dots z_{k'+2M'}} (b(z_1, s'_1) \cdots b(z_{k'}, s'_{k'}) \\
& \quad \times B^*(z_{k'+1}, z_{k'+2}) \cdots B^*(z_{k'+2M'-1}, z_{k'+2M'})) \Big\}^{-1}.
\end{aligned} \tag{22}$$

In the corresponding fermionic formula, Symm is replaced by Asymm.

## 4 SYMBOL OF AN ULTRASECOND-QUANTIZED OPERATOR

First of all, note that the above definition of pseudosymbol does not fully reflect the thermodynamic asymptotics, even though it complies with the Bogolyubov–Dirac rule saying that the creation–annihilation operators in the leading asymptotic term should be replaced by  $c$ -numbers. In any case, we refer to the operator thus defined as the pseudosymbol. Let the operator  $\widehat{H}$  have the form

$$\begin{aligned}
\widehat{H} = & \sum_{l=1}^L \int \cdots \int dx_1 \dots dx_l \widehat{\psi}^+(x_1) \cdots \widehat{\psi}^+(x_l) \\
& \times H_l \left( \overset{2}{x_1}, \dots, \overset{2}{x_l}; -i \frac{\partial}{\partial x_1}, \dots, -i \frac{\partial}{\partial x_l} \right) \widehat{\psi}^-(x_l) \cdots \widehat{\psi}^-(x_1).
\end{aligned} \tag{23}$$

Then, in the case of ultrasecond quantization without creation–annihilation operators  $\widehat{B}^{\pm}(x, y)$  for pairs of particles, the operators  $\overline{\widehat{H}}$  and  $\overline{\widehat{E}}$  defined above satisfy the identity

$$\overline{\widehat{H}} = \overline{\widehat{E}} \overline{\widehat{A}}, \tag{24}$$

where  $\overline{\hat{A}}$  is an operator in  $\mathcal{F}$  of the form

$$\begin{aligned} \overline{\hat{A}} = & \sum_{l=1}^L \sum_{s_1=0}^{\infty} \cdots \sum_{s_l=0}^{\infty} \int \cdots \int dx_1 \cdots dx_l \hat{b}^+(x_1, s_1) \cdots \hat{\psi}^+(x_l, s_l) \\ & \times H_l \left( x_1^2, \dots, x_l^2; -i \frac{\partial}{\partial x_1}, \dots, -i \frac{\partial}{\partial x_l} \right) \hat{b}^-(x_l, s_l) \cdots \hat{b}^-(x_1). \end{aligned} \quad (25)$$

If ultrasecond quantization also takes into account creation–annihilation operators for pairs of particles, then identity (24) remains valid, but the operator  $\overline{\hat{A}}$  has a more complicated form than (25). However, for pairs, the operator  $\overline{\hat{A}}$  has the form

$$\begin{aligned} \overline{\hat{A}} = & \sum_{s=0}^{\infty} \int dx \hat{b}^+(x, s) \left( -\frac{\hbar^2}{2m} \Delta \right) \hat{b}^-(x, s) \\ & + \iint dx dy \hat{B}^+(x, y) \left( -\frac{\hbar^2}{2m} (\Delta_x + \Delta_y) \right) \hat{B}^-(x, y) \\ & + \frac{1}{2} \sum_{s_1=0}^{\infty} \sum_{s_2=0}^{\infty} \iint dx dy V(x, y) \hat{b}^+(x, s_1) \hat{b}^+(y, s_2) \hat{b}^-(y, s_2) \hat{b}^-(x, s_1) \\ & + \sum_{s=0}^{\infty} \iiint dx dy dz (V(x, y) + V(x, z)) \hat{b}^+(x, s) \hat{B}^+(y, z) \hat{B}^-(y, z) \hat{b}^-(x, s) \\ & + \iint dx dy V(x, y) \hat{B}^+(x, y) \hat{B}^-(x, y) \\ & + \frac{1}{2} \iiint dx dy dz dw V(x, y) \hat{B}^+(x, y) \hat{B}^+(z, w) \\ & \times \left( \hat{B}^-(y, w) \hat{B}^-(x, z) + \hat{B}^-(w, y) \hat{B}^-(z, x) \right. \\ & \quad \left. + \hat{B}^-(y, z) \hat{B}^-(w, x) + \hat{B}^-(z, y) \hat{B}^-(x, w) \right). \end{aligned} \quad (26)$$

If we replace the operators  $\hat{B}^{\pm}(x, y)$  in (25) and (26) by  $c$ -numbers, then we obtain the symbol corresponding to the asymptotics in the thermodynamic limit.

## 5 BOSONIC CASE. BOGOLYUBOV FORMULA

Consider a system of  $N$  identical bosons of mass  $m$  in a three-dimensional parallelepiped  $\mathbf{T}$  with side lengths  $L_1$ ,  $L_2$ , and  $L_3$ . We assume that bosons interact with each other and the interaction potential has the form

$$V(N^{1/3}(x - y)), \quad (27)$$

where  $V(\xi)$  is a compactly supported even function and  $x$  and  $y$  are the coordinates of the bosons on  $\mathbf{T}$ . We assume the periodic boundary conditions along  $L_1$  and impose the conditions that the derivatives are zero along  $L_2$ . Note that the interaction potential (27) depends on  $N$  in such a way that its range decreases with increasing particle number  $N$ , while the mean number of particles interacting with a given particle remains constant.

An explicit expression for the ultrasecond-quantized operator  $\widehat{H}$  corresponding to the bosonic system in question under ultrasecond quantization with pairs is given earlier in this paper. As was discussed above, this ultrasecond-quantized operator satisfies the identity

$$\widehat{H} = \widehat{E}\widehat{A}, \quad (28)$$

where  $\widehat{E}$  is the ultrasecond-quantized identity operator and  $\widehat{A}$  is an operator in the ultrasecond quantization space. It is easily seen that the operator

$$\begin{aligned} \widehat{A} = & \iint dx dy \widehat{B}^+(x, y) \left( -\frac{\hbar^2}{2m}(\Delta_x + \Delta_y) + V(N^{1/3}(x - y)) \right) \widehat{B}^-(x, y) \\ & + 2 \iiint dx dy dx' dy' V(N^{1/3}(x - y)) \widehat{B}^+(x, y) \widehat{B}^+(x', y') \widehat{B}^-(x, x') \widehat{B}^-(y, y'), \end{aligned} \quad (29)$$

where  $\widehat{B}^+(x, y)$  and  $\widehat{B}^-(x, y)$  are the bosonic creation-annihilation operators for a pair of particles in the Fock space of ultrasecond quantization, satisfies identity (28). By (28), to find the asymptotics of the spectrum of the bosonic system in question in the limit as  $N \rightarrow \infty$ , one needs to find the corresponding asymptotics for the operator (29).

Since the function (27) multiplied by  $N$  weakly converges as  $N \rightarrow \infty$  to the Dirac delta function, the second term in the operator (29) has the factor  $1/N$  in this limit case. This means that, to find the asymptotics of eigenvalues and eigenfunctions of  $\widehat{A}$ , one can apply the semiclassical methods developed in [9]. The asymptotics of eigenvalues and eigenfunctions is determined by the symbol of the operator (29); this symbol is called the *true symbol* of the ultrasecond-quantized problem. The true symbol corresponding to the operator (29) is given by the following functional defined for a pair of functions  $\Phi^+(x, y)$  and  $\Phi(x, y)$ :

$$\begin{aligned} \mathcal{H}[\Phi^+(\cdot), \Phi(\cdot)] = & \iint dx dy \Phi^+(x, y) \left( -\frac{\hbar^2}{2m}(\Delta_x + \Delta_y) \right) \Phi(x, y) \\ & + 2 \iiint dx dy dx' dy' (NV(N^{1/3}(x - y))) \Phi^+(x, y) \\ & \times \Phi^+(x', y') \Phi(x, x') \Phi(y, y'). \end{aligned} \quad (30)$$

Since the number of particles is preserved in the system for the functions  $\Phi^+(x, y)$  and  $\Phi(x, y)$ , we obtain the condition

$$\iint dx dy \Phi^+(x, y) \Phi(x, y) = \frac{1}{2}. \quad (31)$$

By the asymptotic methods in [9], to each solution of the system

$$\Omega\Phi(x, y) = \frac{\delta\mathcal{H}}{\delta\Phi^+(x, y)}, \quad \Omega\Phi^+(x, y) = \frac{\delta\mathcal{H}}{\delta\Phi(x, y)} \quad (32)$$

with condition (31), there corresponds an asymptotic series of eigenfunctions and eigenvalues of the operator (29) in the limit as  $N \rightarrow \infty$ . It follows from the explicit form of the true symbol (30) that system (32) can be represented in the form

$$\begin{aligned} \Omega\Phi(x, y) &= -\frac{\hbar^2}{2m}(\Delta_x + \Delta_y)\Phi(x, y) \\ &\quad + \iint dx' dy' (NV(N^{1/3}(x-y)) + NV(N^{1/3}(x'-y'))) \\ &\quad \times \Phi^+(x', y')\Phi(x, x')\Phi(y, y'), \\ \Omega\Phi^+(x, y) &= -\frac{\hbar^2}{2m}(\Delta_x + \Delta_y)\Phi^+(x, y) \\ &\quad + 2 \iint dx' dy' (NV(N^{1/3}(x-x')) + NV(N^{1/3}(y-y'))) \\ &\quad \times \Phi(x', y')\Phi^+(x, x')\Phi^+(y, y'). \end{aligned} \quad (33)$$

Let  $v_q$  be the coefficient in the Fourier series expansion of the potential  $NV(N^{1/3}x)$  on the parallelepiped  $(L_1, L_2, L_2)$ :

$$v_q = \frac{1}{L_1 L_2^2} \int_T e^{-iqx} NV(\sqrt[3]{N}(x)) dx, \quad v_{-q} = v_q. \quad (34)$$

The exact solution of system (33) is given by the functions

$$\Phi_{k_1, k_2}^+ = \frac{1}{L_1 L_2^2} e^{-ik_1(x+y)} \cos(k_2(x-y)), \quad (35)$$

$$\Phi_{k_1, k_2} = \frac{1}{L_1 L_2^2} \sum_l \varphi_{k_2, l} e^{ik_1(x+y)} e^{il(x-y)}, \quad (36)$$

where  $k_1 = (n_1/L_1, 0, 0)$  and  $k_2 = (0, n_2/L_2, n_3/L_2)$ , with the eigenvalue

$$\Omega = \frac{\hbar^2}{m}(k_1^2 + k_2^2) + v_0 + v_{2k_2}, \quad (37)$$

where the function  $\varphi_{k_2, l}$  has the form

$$\begin{aligned} \varphi_{k_2, l} &= -\frac{b_l}{2} + \frac{1}{2}\sqrt{b_l^2 - 1}, \quad l^2 > k_2^2, \\ \varphi_{k_2, l} &= -\frac{b_l}{2} - \frac{1}{2}\sqrt{b_l^2 - 1}, \quad l^2 < k_2^2, \\ \varphi_{k_2, k_2} &= \frac{1}{2}, \quad \varphi_{k_2, l} = \varphi_{k_2, -l}, \\ b_l &= \frac{\hbar^2/m(l^2 - k_2^2) - (v_0 + v_{2k_2})}{v_{l-k_2} + v_{l+k_2}}, \quad b_l = b_{-l}. \end{aligned} \quad (38)$$

(If  $v_l \rightarrow 0$ , then  $\varphi_{k_2,l} \rightarrow 0$ .)

The pair  $(k_1, k_2)$  of vectors plays the role of parameters numbering various solutions of this system. The vector  $\hbar k_1/m$  is equal to the bosonic system flow velocity along the capillary. The vector  $k_2$  is the wave vector of the transverse mode.

Note that  $b_l \rightarrow \infty$  as  $|l| \rightarrow \infty$ , since

$$|v_l| = \frac{1}{L_1 L_2^2} \int_{NT} e^{-il\xi/N} V(\xi) d\xi \leq \frac{1}{L_1 L_2^2} \int_{NT} |V(\xi)| d\xi < \frac{1}{L_1 L_2^2} \int_{R^3} |V(\xi)| d\xi, \quad (39)$$

and consequently

$$\varphi_{k,l} \cong \frac{1}{b_l^2}, \quad (40)$$

whence it follows that the series (36) converges absolutely.

We split the series (36) into two parts, one with  $l \leq N^{1/6}$  and the other with  $l > N^{1/6}$ . The first part of the sum converges as  $N \rightarrow \infty$  modulo  $O(N^{-1/6})$  to

$$\begin{aligned} b_l &\rightarrow \frac{\hbar^2(l^2 - k_2^2)}{2mV_0} - 1 \stackrel{\text{def}}{=} b_l^0, \\ \varphi_{k_2,l} &\rightarrow -\frac{b_l}{2} \pm \frac{1}{2} \sqrt{b_l^2 - 1} \stackrel{\text{def}}{=} \varphi_{k_2,l}^0. \end{aligned} \quad (41)$$

This readily follows from the change of variables  $\sqrt[3]{N}x = \xi$  in (34).

The second part of the sum converges to zero as  $O(N^{-1/6})$  by (40). Hence system (33) supplemented with condition (31) has the following family of solutions for  $k_1 = 0$  in the limit as  $N \rightarrow \infty$ :

$$\begin{aligned} \Phi_k^+(x, y) &= \frac{1}{L_1 L_2^2} \cos(k(x - y)), \\ \Phi_k(x, y) &= \frac{1}{L_1 L_2^2} \sum_l \varphi_{k,l} \exp(il(x - y)), \end{aligned} \quad (42)$$

where  $k$  and  $l$  are three-dimensional vectors of the form

$$2\pi \left( 0, \frac{n_2}{L_2}, \frac{n_3}{L_2} \right),$$

$n_2$  and  $n_3$  are integers, the  $\varphi_{k,l}$  in (28) have the form

$$\varphi_{k_2,l}^0 = \frac{1}{2V_0} \left( \frac{\hbar^2}{2m} (k_2^2 - l^2) + V_0 \pm \sqrt{\left( \frac{\hbar^2}{2m} (k_2^2 - l^2) + V_0 \right)^2 - V_0^2} \right), \quad (43)$$

(the plus sign is taken for  $l^2 > k_2^2$ , and the minus sign, for  $l^2 < k_2^2$ ), and finally  $V_0$  stands for the expression

$$V_0 = \frac{1}{L_1 L_2^2} \int dx V(x), \quad (44)$$

the integral being taken over  $\mathbf{R}^3$ . The vector  $k$  in (42) plays the role of a parameter numbering various solutions of system (33), (31). The solutions (42) are standing waves; there is no flow in the corresponding series.

The leading asymptotic term of the eigenvalues for the series corresponding to the solution (35), (36) is equal to  $N$  times the value of the symbol (30) on these functions:

$$E_{k_1, k_2} = N \left( \frac{\hbar^2(k_1^2 + k_2^2)}{2m} + \frac{V_0}{2} \right). \quad (45)$$

The asymptotics of eigenvalues and eigenfunctions (in particular, the terms following  $E_{k_1, k_2}$ ) is determined not only by system (33) but also by the solutions of the variational system corresponding to the Hamiltonian system. The variational system for (33) has the form

$$\begin{aligned} (\Omega - \lambda)F(x, y) &= -\frac{\hbar^2}{2m}(\Delta_x + \Delta_y)F(x, y) \\ &\quad + 2N \iint dx' dy' (V(\sqrt[3]{N}(x - y)) + V(\sqrt[3]{N}(x' - y'))) \\ &\quad \times (G(x', y')\Phi(x, x')\Phi(y, y') + \Phi^+(x', y')F(x, x')\Phi(y, y') \\ &\quad + \Phi^+(x', y')\Phi(x, x')F(y, y')), \\ (\Omega + \lambda)G(x, y) &= -\frac{\hbar^2}{2m}(\Delta_x + \Delta_y)G(x, y) \\ &\quad + 2N \iint dx' dy' (V(\sqrt[3]{N}(x - x')) + V(\sqrt[3]{N}(y - y'))) \\ &\quad \times (F(x', y')\Phi^+(x, x')\Phi^+(y, y') + \Phi(x', y')G(x, x')\Phi^+(y, y') \\ &\quad + \Phi(x', y')\Phi^+(x, x')G(y, y')). \end{aligned} \quad (46)$$

To find the spectrum of quasiparticles, one should select solutions of the variational system satisfying the selection rule [10] for the complex germ for self-adjoint operators with real spectrum.

Consider the case in which  $k_2 \neq 0$ . We substitute the solutions (35), (36) into (46) and take into account symmetry. Then we see that the solutions of

the variational system have the form

$$\begin{aligned}
G_l(x, y) &= u_{1,l}(\exp(i(-k_1 + k_2)x + i(-k_1 + l)y) + \exp(i(-k_1 + k_2)y + i(-k_1 + l)x)) \\
&\quad + u_{2,l}(\exp(i(-k_1 - k_2)x + i(-k_1 + 2k_2 + l)y) \\
&\quad + \exp(i(-k_1 - k_2)y + i(-k_1 + 2k_2 + l)x)), \\
F_l(x, y) &= -v_{1,l}(\exp(i(k_1 + k_2)x + i(k_1 + l)y) + \exp(i(k_1 + k_2)y + i(k_1 + l)x)) \\
&\quad - v_{2,l}(\exp(i(k_1 - k_2)x + i(k_1 + 2k_2 + l)y) \\
&\quad + \exp(i(k_1 - k_2)y + i(k_1 + 2k_2 + l)x)) \\
&\quad + \sum_{l' \neq l, l+2k_2} w_{l,l'}(\exp(i(k_1 + k_2 + l - l')x + i(k_1 + l')y) \\
&\quad + \exp(i(k_1 + k_2 + l - l')y + i(k_1 + l')x)),
\end{aligned} \tag{47}$$

where  $l \neq -k_2$  and the numerical coefficients  $u_{1,l}$ ,  $u_{2,l}$ ,  $v_{1,l}$ ,  $v_{2,l}$ , and  $w_{l,l'}$  are determined from an infinite system of equations. This system contains a closed subsystem of four equations for the coefficients  $u_{1,l}$ ,  $u_{2,l}$ ,  $v_{1,l}$ , and  $v_{2,l}$ , which can be rewritten in the form

$$\tilde{\lambda}X = MX, \tag{48}$$

where

$$\tilde{\lambda} = \lambda + \frac{\hbar^2}{m}k_1(k_2 + l)$$

and  $X$  is the column vector

$$X = \begin{pmatrix} u_{1,l} \\ u_{2,l} \\ v_{1,l} \\ v_{2,l} \end{pmatrix}.$$

Equations (47) and (48) give the matrix  $M$

$$M = \begin{pmatrix} B_l + \frac{v_{l-k_2}}{2} & \frac{v_{2k_2} + v_{l+k_2}}{2} & -\frac{v_{l+k_2} + v_{l-k_2}}{2} & 0 \\ \frac{v_{2k_2} + v_{l+k_2}}{2} & B_{l+2k_2} + \frac{v_{l+3k_2}}{2} & 0 & -\frac{v_{l+k_2} + v_{l+3k_2}}{2} \\ 2(v_0 + v_{l-k_2})\varphi_{k_2,l} & (v_{2k_2} + v_{l+k_2}) \times (\varphi_{k_2,l} + \varphi_{k_2,l+2k_2}) & -B_l - \frac{v_{l-k_2}}{2} & -\frac{v_{2k_2} + v_{l+k_2}}{2} \\ (v_{2k_2} + v_{l+k_2}) \times (\varphi_{k_2,l} + \varphi_{k_2,l+2k_2}) & 2(v_0 + v_{l+3k_2})\varphi_{k_2,l+2k_2} & -\frac{v_{2k_2} + v_{l+k_2}}{2} & -B_{l+2k_2} - \frac{v_{l+3k_2}}{2} \end{pmatrix},$$

where the  $B_l$  have the form

$$B_l = \frac{\hbar^2}{2m}(l^2 - k_2^2) + (v_{l-k_2} + v_{l+k_2})\varphi_{k_2,l} - \frac{v_{2k_2}}{2}.$$

Obviously,

$$v_{l+k_2} + v_{l+3k_2} = v_{l-k_2} + v_{l+k_2} + O\left(\frac{1}{N}\right) \tag{49}$$



uniformly in  $l$  as  $N \rightarrow \infty$  (it suffices to make the change of variables (39)). Then the matrix  $M$  can be approximately represented as the block matrix

$$M = \begin{pmatrix} C & -V_l E \\ D & -C \end{pmatrix},$$

where  $E$  is the identity  $2 \times 2$  matrix and

$$V_l = \frac{v_{l-k_2} + v_{l+k_2}}{2}.$$

We also introduce

$$V_l^+ = \frac{v_{l+k_2} + v_{2k_2}}{2}, \quad V_l^- = \frac{v_{l-k_2} + v_0}{2}.$$

The eigenvalue corresponding to Eq. (48) have the form

$$\begin{aligned} \lambda_{k_1, k_2, l} = & -2ak_1(k_2 + l) \\ & \pm \left( \frac{1}{2}(a(l^2 - k_2^2) + V_l - V_l^+)^2 + \frac{1}{2}(a(l_1^2 - k_2^2) + V_l - V_l^+)^2 + V_l^{+2} - V_l^2 \right. \\ & \left. \pm \frac{1}{2}(a(l_1^2 + l^2 - 2k_2^2) + 2V_l - 2V_l^+) \sqrt{a^2(l_1^2 - l^2)^2 + 4V_l^{+2}} \right)^{1/2}, \end{aligned} \quad (50)$$

where

$$a = \frac{\hbar^2}{2m}, \quad l_1 = l + 2k_2.$$

Before passing to the limit, one has

$$\begin{aligned} \lambda_{k_1, k_2, l} = & -2ak_1 l \pm \left( (al^2 + V_l - V_l^+)^2 + V_l^{+2} - V_l^2 \pm 2(al^2 + V_l - V_l^+) |V_l^+| \right)^{1/2}, \\ & V_l = v_l, \quad V_l^+ = V_l^- = \frac{v_l + v_0}{2} \end{aligned}$$

for  $k_2 = 0$ . By formally setting  $k_2 = 0$ , we arrive at Bogolyubov's well-known formula

$$\lambda_{1, l} = -\frac{\hbar^2}{m} k_1 l + \sqrt{\left( \frac{\hbar^2 l^2}{2m} + v_l \right)^2 - v_l^2}.$$

Here  $v_l$  is the Fourier transform of the potential.

(We assume that  $L_2$  is much larger than some standard length, say, the electron radius  $r_0$ , and that although  $L_1 \gg L_2$ , we can take a sufficiently large integer  $n_1$ . In other words,  $L_2/r_0 \rightarrow \infty$  and  $L_1/r_0 \rightarrow \infty$ , but the vector  $k_1 = (n_1/L_1, 0, 0)$  remains finite, since  $n_1 \rightarrow \infty$ .)

In the language of nonstandard analysis, this means that  $L_2$  is an infinite (nonstandard) number,  $L_1$  and  $n_1$  are nonstandard numbers of higher order, and  $k_1 = (n_1/L_1, 0, 0)$  is a standard finite number.

Then  $k_2$  is equal to an infinitesimal nonstandard zero,  $k_2 \cong 0$ , and  $k_1$  is a standard number.)

## 6 CASE OF A FERMI FLUID

Consider the Hamiltonian system for fermions:

$$\begin{aligned}
\Omega\Phi(x, y) &= \left(-\frac{\hbar^2}{2m}(\Delta_x + \Delta_y)\right)\Phi(x, y) \\
&\quad + 2N \iint dx' dy' (V(x - y) + V(x' - y'))\Phi^+(x', y')\Phi(x, x')\Phi(y', y), \\
\Omega\Phi^+(x, y) &= \left(-\frac{\hbar^2}{2m}(\Delta_x + \Delta_y)\right)\Phi^+(x, y) \\
&\quad + 2N \iint dx' dy' (V(x - x') + V(y - y'))\Phi(x', y')\Phi^+(x, x')\Phi^+(y', y).
\end{aligned} \tag{51}$$

The functions  $\Phi^+(x, y)$  and  $\Phi(x, y)$  are antisymmetric and satisfy the normalization condition

$$\iint dx dy \Phi^+(x, y)\Phi(x, y) = \frac{1}{2}. \tag{52}$$

Let us represent the interaction potential by a Fourier series:

$$NV(x) = \sum_p v_p e^{ipx}, \quad v_p = \frac{1}{L_1 L_2^2} \int dx NV(x) e^{-ipx}, \quad v_p = v_{-p}.$$

We seek the solution of system (51), (52) in the form

$$\begin{aligned}
\Phi_{k_1, k_2}^+(x, y) &= \frac{1}{L_1 L_2^2} e^{-ik_1(x+y)} \sin(k_2(x - y)), \\
\Phi_{k_1, k_2}(x, y) &= \frac{1}{L_1 L_2^2} \sum_l \varphi_{k_2, l} e^{il(x-y) + ik_1(x+y)},
\end{aligned} \tag{53}$$

where  $k_1$ ,  $k_2$ , and  $l$  are three-dimensional vectors of the form

$$2\pi \left( \frac{n_1}{L_1}, \frac{n_2}{L_2}, \frac{n_3}{L_2} \right)$$

and  $n_1$ ,  $n_2$ , and  $n_3$  are integers. The numbers  $\varphi_{k_2, l}$  should satisfy the condition

$$\varphi_{k_2, l} = -\varphi_{k_2, -l}.$$

After the substitution, we find that the eigenvalue is equal to

$$\Omega = \frac{\hbar^2}{m}(k_1^2 + k_2^2) + v_{2k_2} - v_0$$

and the  $\varphi_{k_2, l}$  have the form

$$\varphi_{k_2, l} = -\frac{ib_{k_2, l}}{2} \pm \frac{1}{2} \sqrt{1 - b_{k_2, l}^2}, \quad b_{k_2, l} \equiv \frac{(\hbar^2/m)(l^2 - k_2^2) + (v_0 - v_{2k_2})}{v_{l-k_2} - v_{l+k_2}}.$$

Note that

$$b_{k_2, l} = -b_{k_2, -l}.$$

Set

$$\varphi_{k_2, l} = -\frac{ib_{k_2, l}}{2} + \frac{1}{2} \frac{v_{l-k_2} - v_{l+k_2}}{|v_{l-k_2} - v_{l+k_2}|} \sqrt{1 - b_{k_2, l}^2}.$$

Then  $\varphi_{k_2, l}$  will be equal to  $-\varphi_{k_2, -l}$ .

Consider the fermionic variational system

$$\begin{aligned} (\Omega - \lambda)F(x, y) &= -\frac{\hbar^2}{2m}(\Delta_x + \Delta_y)F(x, y) \\ &\quad + 2N \iint dx' dy' (V(\sqrt[3]{N}(x - y)) + V(\sqrt[3]{N}(x' - y'))) \\ &\quad \times (G(x', y')\Phi(x, x')\Phi(y', y) + \Phi^+(x', y')F(x, x')\Phi(y', y) \\ &\quad + \Phi^+(x', y')\Phi(x, x')F(y', y)), \\ (\Omega + \lambda)G(x, y) &= -\frac{\hbar^2}{2m}(\Delta_x + \Delta_y)G(x, y) \\ &\quad + 2N \iint dx' dy' (V(\sqrt[3]{N}(x - x')) + V(\sqrt[3]{N}(y - y'))) \\ &\quad \times (F(x', y')\Phi^+(x, x')\Phi^+(y', y) + \Phi(x', y')G(x, x')\Phi^+(y', y) \\ &\quad + \Phi(x', y')\Phi^+(x, x')G(y', y)). \end{aligned} \tag{54}$$

Its solutions have the form

$$\begin{aligned} G_l(x, y) &= u_{1, l}(e^{i(-k_1+k_2)x+i(-k_1+l)y} - e^{i(-k_1+k_2)y+i(-k_1+l)x}) \\ &\quad + u_{2, l}(e^{i(-k_1-k_2)x+i(-k_1+2k_2+l)y} - e^{i(-k_1-k_2)y+i(-k_1+2k_2+l)x}), \\ F_l(x, y) &= v_{1, l}(e^{i(k_1+k_2)x+i(k_1+l)y} - e^{i(k_1+k_2)y+i(k_1+l)x}) \\ &\quad + v_{2, l}(e^{i(k_1-k_2)x+i(k_1+2k_2+l)y} - e^{i(k_1-k_2)y+i(k_1+2k_2+l)x}) \\ &\quad + \sum_{l' \neq l, l+2k_2} w_{l, l'}(e^{i(k_1+k_2+l-l')x+i(k_1+l')y} - e^{i(k_1+k_2+l-l')y+i(k_1+l')x}), \end{aligned} \tag{55}$$

where  $l \neq -k_2$  and the numerical coefficients  $u_{1, l}$ ,  $u_{2, l}$ ,  $v_{1, l}$ ,  $v_{2, l}$ , and  $w_{l, l'}$  are determined from an infinite system of equations. This system contains a closed subsystem of four equations for the coefficients  $u_{1, l}$ ,  $u_{2, l}$ ,  $v_{1, l}$ , and  $v_{2, l}$ , which can be rewritten in the standard form (48).

The matrix  $M$  is given by

$$M = \begin{pmatrix} B_l + \frac{v_{l-k_2}}{2} & \frac{v_{l+k_2} - v_{2k_2}}{2} & \frac{v_{l-k_2} - v_{l+k_2}}{2} & 0 \\ \frac{v_{l+k_2} - v_{2k_2}}{2} & B_{l+2k_2} + \frac{v_{l+3k_2}}{2} & 0 & \frac{v_{l+3k_2} - v_{l+k_2}}{2} \\ 2i(v_{l-k_2} - v_0)\varphi_{k_2, l} & i(v_{2k_2} - v_{l+k_2}) \times (\varphi_{k_2, l+2k_2} - \varphi_{k_2, l}) & -B_l - \frac{v_{l-k_2}}{2} & -\frac{v_{l+k_2} - v_{2k_2}}{2} \\ i(v_{2k_2} - v_{l+k_2}) \times (\varphi_{k_2, l+2k_2} - \varphi_{k_2, l}) & 2i(v_0 - v_{l+3k_2})\varphi_{k_2, l+2k_2} & -\frac{v_{l+k_2} - v_{2k_2}}{2} & -B_{l+2k_2} - \frac{v_{l+3k_2}}{2} \end{pmatrix},$$

and the  $B_l$  have the form

$$B_l = \frac{\hbar^2}{2m}(l^2 - k_2^2) + i(v_{l+k_2} - v_{l-k_2})\varphi_{k_2,l} - \frac{v_2 k_2}{2}.$$

For  $k_2 = 0$ , the eigenvalues have the form

$$\lambda_{1,l} = -\frac{\hbar^2}{m}lk_1 + \frac{\hbar^2 l^2}{2m}, \quad \lambda_{2,l} = -\frac{\hbar^2}{m}lk_1 + \left| \frac{\hbar^2 l^2}{2m} + v_l - v_0 \right|.$$

This means that we choose the vectors  $k_2 = (0, 1/L_2, 0)$  and  $k_1 = (n/L_1, 0, 0)$ . If  $n \gg 1$  and hence  $k_1 \gg k_2$ , then the fluid velocity vector is mainly directed along the tube. As before, let  $L_1 = \infty$ ,  $L_2 = \infty$ , and  $n = \infty$  be nonstandard numbers, let  $L_1 \gg L_2$ , and let  $k_1$  be a standard number. Then  $k_2 \cong 0$ , and  $\varphi_{k_2,l}$  takes nonstandard values for  $k_2 \cong 0$ . However, if  $0 \leq l \leq \sqrt{k_2}M$ , where  $M \leq \infty$  (i.e.,  $0 \leq l \leq \infty$ ), then  $\varphi_{k_2,l}$  can be assumed to be a standard number. For  $k_2 \cong 0$ , taking into account the selection rule, we obtain the formula

$$\lambda_{2,l} = -\frac{\hbar^2}{m}lk_1 + \left| \frac{\hbar^2 l^2}{2m} + v_l - v_0 \right|$$

by analogy with the bosonic case and formula (1).

As was mentioned in Sec. 2, the behavior of the expression under the modulus sign is similar to that of the Landau curve for bosons. The criterion for  $k_1$  has the form

$$|k_1| \leq \frac{m}{\hbar^2} \min_l \left| \frac{v_l - v_0}{|l|} + \frac{\hbar^2 l}{2m} \right|,$$

similar to the Landau criterion (the vapor-destroying velocity). We set the parameter  $k_2$  to be an infinitesimal nonstandard number. However, it is not exactly zero, and the presence of a nonzero  $k_2$  results in a spectral gap. To compute the gap in the first approximation, one has to find the spectrum of the matrix  $M$  modulo  $O(k_2^2)$ . Then, by using the selection rules, one can determine whether there is a gap in the spectrum.

## ACKNOWLEDGMENTS

The author wishes to express gratitude to D. S. Golikov for re-calculation and verification of all formulas.

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